JOURNAL OF APPROXIMATION THEORY **40**, 364–374 (1984)

Chebyshev Centers in Normed Spaces

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Communicated by Oved Shisha

Received September 13, 1982; revised February 7, 1983

Let E be a normed linear space, F a bounded subset, Y a closed subset of E. A nonnegative real number $r_Y(F)$ is called the relative Chebyshev radius of F with respect to Y if $r_Y(F)$ is the infimum of all numbers $r > 0$ for which there exists a $y \in Y$ such that F is contained in the closed ball $B(y, r)$ with center y and radius r. Any point $y \in Y$ for which $F \subset B(y, r_Y(F))$ is called a relative Chebyshev center of F with respect to Y. We denote the set of all relative Chebyshev centers of F with respect to Y by $z_Y(F)$.

In this paper we investigate several questions concerning characterization and existence of relative Chebyshev centers, and the continuity of the Chebyshev center map. In Section 1 we give a formula for the relative Chebyshev radius of a bounded set F with respect to Y in terms of the relative radius of F with respect to hyperplanes from the annihilator of Y. For F totally bounded this formula was obtained in [8]. Let F be a bounded set which is contained in the closed ball $B(y, r)$, where $r = r(y, F) \equiv \sup\{\|x - y\|; x \in F\}$. In Section 2 we are looking for necessary and sufficient conditions for $B(y, r)$ to be the Chebyshev ball of F. For Hilbert space, a characterization was given in [5]. However, the necessity part of this characterization requires a property valid only for F compact. We give necessary and sufficient conditions for both the compact and noncompact case and deduce several corollaries. In Section 3 we show that every infinite dimensional normed space E has an equivalent norm such that $c_0(E)$ does not admit relative centers for all pairs of points in $l_\infty(E)$. In Section 4 we investigate the Lipschitz constants of the Chebyshev center map restricted to certain families of "admissible" pairs of sets, as introduced in [5].

0021-9045/84 $3.00

In Sections 5 and 6, finally, we discuss the upper semicontinuity of the Chebyshev center map and the proximinality of isomorphic images of proximinal subspaces.

1. A Duality Theorem for Global Approximation

Proposition 1.1 generalizes a theorem by Franchetti and Cheney ([8], Theorem 2.2).

1.1. Proposition. *For every linear subspace Y of a normed space E and for every bounded subset F of E, we have*

$$r_Y(F) = \max_{f \in Y^{\perp}} r_{f^{-1}(0)}(F).$$

Proof. For every $f \in Y^{\perp}$, $r_{f^{-1}(0)}(F) \leqslant r_Y(F)$ (since $f^{-1}(0) \supset Y$). If $r_Y(F) = r(F)$ then for every $f \in Y^{\perp}$ we have $r(F) = r_Y(F) \geqslant r_{f^{-1}(0)}(F) \geqslant r(F)$, hence equality. If $r_Y(F) > r(F)$, then every $x \in E$ with $r(x, F) < r_Y(F)$ is an interior point of $K = \bigcap_{x \in F} B(x, r_Y(F))$. We can apply the Hahn–Banach theorem and extend Y to a hyperplane $H = f^{-1}(0)$, $f \in Y^{\perp}$, with $H \cap K^0 = \varnothing$, i.e., with $r_H(F) \geqslant r_Y(F)$.

Theorem 2.2 in [8] assumed F to be totally bounded, and the proof used finite-dimensional approximants for F.

2. Characterization of Chebyshev Centers

Proposition 2 in [5] states that in Hilbert space, if a closed, bounded, and convex subset K is contained in a closed ball $B(x, r)$, then the ball is the Chebyshev ball for K if and only if $x \in \operatorname{conv}(K \cap S(x, r))$. The proof of the necessity part assumes that the distance between the sphere $S(x, r) = \{y \in E; \|x - y\| = r\}$ and the "half" of K disjoint with it is positive and is, therefore, valid only in the compact case.

2.1. Example. In the Hilbert sequence space l_2, with the orthonormal basis $(e_n)_{n=1}^{\infty}$, let $K = \overline{\operatorname{conv}}\{(1 - 1/n)\, e_n;\ n = 1, 2, \ldots\}$. The unit ball $B(0, 1)$ is the Chebyshev ball for K, but $K \cap S(0, 1) = \varnothing$ (a common point must be extremal in K, hence, $(1 - 1/n)\, e_n$ for some n).

In the compact case the proposition can be generalized to relative centers in general normed spaces.

2.2. Proposition. *Let Y be a convex subset and K a compact subset of* *the normed space E,* $y_0 \in Y$, $r_0 = r(y_0, K)$. *Then* $y_0 \in z_Y(K)$ *if and only if* $y_0 \in z_Y(K \cap S(y_0, r_0))$.

Proof. Without loss of generality, we may assume $y_0 = 0$, $r_0 = 1$. Denote $S = S_E = S(0, 1)$, the unit sphere of the space E. Since K is compact, $K \cap S \neq \varnothing$. If $0 \in z_Y(K \cap S)$, then $1 \geqslant r_Y(K) \geqslant r_Y(K \cap S) \geqslant 1$, hence $0 \in z_Y(K)$. If $r_Y(K) = 1$ but $0 \notin z_Y(K \cap S)$, there are $y \in Y$ and $\delta > 0$ such that $r \equiv r(y, K \cap S) < 1 - \delta$. Let $K_1 \equiv \{x \in K;\ \|x - y\| \geqslant 1 - \delta\}$. Then $K_1 \cap S \neq \varnothing$ and, by compactness of K_1, there is $d > 0$ with $\|x\| \leqslant 1 - d$ for all $x \in K_1$. Let $\varepsilon > 0$ be smaller than $\min(1, d/\|y\|)$. For $x \in K_1$ we have $\|x - \varepsilon y\| \leqslant \|x\| + \varepsilon \|y\| < 1 - d + \varepsilon \|y\|$, while for $x \in K \backslash K_1$ we have $\|x - \varepsilon y\| \leqslant (1 - \varepsilon)\|x\| + \varepsilon \|x - y\| < 1 - \delta\varepsilon$. Thus $\varepsilon y \in Y$ satisfies $r(\varepsilon y, K) \leqslant \max(1 - d + \varepsilon \|y\|, 1 - \delta\varepsilon) < 1 = r_Y(K)$, which is impossible. Therefore, $0 \in z_Y(K \cap S)$.

Remark. It suffices that K be "M-compact" in the sense of Panda and Kapoor [12], i.e., that whenever $k_n \in K$ and $\|y - k_n\| \to r(y, K)$ there be a convergent subsequence $k_{n_i} \to k \in K$.

The analogue for the noncompact case is

2.3. Proposition. *Let Y be a convex subset and F a bounded subset of the normed space E,* $y_0 \in Y$, $r_0 = r(y_0, F)$. *Then* $y_0 \in z_Y(F)$ *if and only if* $y_0 \in z_Y(F \backslash B(y_0, t))$ *for some (or all)* $0 \leqslant t < r_0$.

Proof. Again assume $y_0 = 0$, $r_0 = 1$. If $0 \in z_Y(F \backslash tB)$ then, clearly, $1 = r(0, F) = r(0, F \backslash tB) = r_Y(F \backslash tB) \leqslant r_Y(F) = 1$, hence, $0 \in z_Y(F)$. If $r_Y(F) = 1$ but $0 \notin z_Y(F \backslash tB)$, there are $0 \neq y \in Y$, $d > 0$ such that $r(y, F \backslash tB) < 1 - d$. Let $0 < \varepsilon < \min(1, (1 - t)/\|y\|)$. If $x \in tB$ then $\|x - \varepsilon y\| \leqslant \|x\| + \varepsilon \|y\| = t + \varepsilon \|y\|$, while if $x \in F \backslash tB$ then $\|x - \varepsilon y\| \leqslant (1 - \varepsilon)\|x\| + \varepsilon \|x - y\| < 1 - \varepsilon\delta$, thus $r(\varepsilon y, F) < 1$ and $0 \notin z_Y(F)$.

Proposition 2.3 had been observed by Franchetti (unpublished). From Proposition 2.2 we deduce the correct version of the Borwein–Keener statement, in a slightly generalized form

2.4. Corollary. *If Y is a closed linear subspace and K is a compact subset of a Hilbert space,* $y_0 \in Y$ *and* $r_0 = r(y_0, K)$, *then* $y_0 = z_Y(K)$ *if and only if* $y_0 \in \overline{\text{conv}}\, P_Y(K \cap S(y_0, r_0))$ *(where* P_Y *is the metric projection onto Y).*

Proof. Again we may assume $y_0 = 0$, $r_0 = 1$, and we have to show that $0 = z_Y(K)$ if and only if $0 \in \overline{\text{conv}}\, P_Y(K \cap S)$.

The "only if part" follows immediately from Proposition 2.2 and the fact that in Hilbert space $z_Y(F) \in \text{conv}\, P_Y(F)$ [13, Theorem 3.3]. For the "if" part, let $0 \neq y \in Y$. If $(P_Y x, y) > 0$ for all $x \in K \cap S$ then, by compactness and continuity of P_Y, $(P_Y(K \cap S), y) \geqslant \varepsilon > 0$ which is impossible since $0 \in \overline{\text{conv}}\, P_Y(K \cap S)$. Thus $(P_Y x, y) \leqslant 0$ for some $x \in K \cap S$ and, therefore,

$$\begin{aligned} r(y, K)^2 &\geqslant \|x - y\|^2 = \|x - P_Y x\|^2 + \|P_Y x - y\|^2 \\ &\geqslant \|x - P_Y x\|^2 + \|P_Y x\|^2 + \|y\|^2 = 1 + \|y\|^2 > 1 = r(0, K)^2. \end{aligned}$$

Since y was any point in $Y \backslash \{0\}$, $0 = z_Y(K)$.

The noncompact analogue of Corollary 2.4 is

2.5. COROLLARY. *If Y is a closed linear subspace and F is a bounded subset of a Hilbert space, $y_0 \in Y$ and $r_0 = r(y_0, F)$, then $y_0 \in z_Y(F)$ if and only if there exists a $t_0 < r_0$ such that $y_0 \in \overline{\text{conv}}\, P_Y(F \backslash B(y_0, t))$ for every $t_0 \leqslant t < r_0$.*

Proof. Again we may assume $y_0 = 0$, $r_0 = 1$, and we have to show that $0 = z_Y(F)$ iff $0 \in \bigcap_{t_0 \leqslant t < 1} \overline{\text{conv}}\, P_Y(F \backslash B(0, t))$ for some $t_0 < 1$. The "only if" part follows immediately from Proposition 2.3 and the fact that in Hilbert space $z_Y(A) \in \text{conv}\, P_Y(A)$. For the "if" part, let $0 \neq y \in Y$. Let $0 < 1 - t_0 < \|y\|^2/4$, $t \geqslant t_0$. Since $0 \in \text{conv}\, P_Y(F \backslash tB)$, there is an $x \in P_Y(F \backslash tB)$ such that $(P_Y x, y) < 1 - t$. But then

$$\begin{aligned} \|x - y\|^2 &= \|x - P_Y x\|^2 + \|P_Y x - y\|^2 \\ &> \|x - P_Y x\|^2 + \|P_Y x\|^2 + \|y\|^2 - 2(1 - t) \\ &= \|x\|^2 + \|y\|^2 - 2(1 - t) > t^2 + \|y\|^2 - 2(1 - t) > 1. \end{aligned}$$

It follows that $0 \in z_Y(F)$.

Remark. If we assume Y only to be a closed convex set, then the analogue of Corollary 2.4 need not hold, e.g., in the Euclidean plane let Y be the unit disk, $K = \{(2, 2), (2, -2)\}$. Then $z_Y(K) = (1, 0) \notin \overline{\text{conv}}\, P_Y K = \frac{1}{2}\sqrt{2}\,[(1, -1), (1, 1)]$.

Another corollary, in general normed spaces, is the following generalization of a result of Franchetti and Cheney ([8, Theorem 2.1a]).

2.6. COROLLARY. *Let Y be a convex subset and K be a compact subset of the normed space E, $y_0 \in Y$ and $r_0 = r(y_0, K)$. Then $y_0 \in z_Y(K)$ if and only if for every $y \in Y$ there are $x \in K$ and $\varphi \in \text{ext}\, B^*$ such that $\varphi(x - y) \geqslant \varphi(x - y_0) \geqslant r_0$.*

Proof. Sufficiency is obvious, since in this case, for all $y \in Y$,

$$\sup\{\|z - y\|; z \in K\} \geqslant \|x - y\| \geqslant \varphi(x - y) \geqslant r_0.$$

For necessity, we may assume $y_0 = 0 \in z_Y(K)$, $r_0 = 1$, $y \in Y$. By the proposition, $r(y, K \cap S) \geqslant 1$, so that for some $x_0 \in K \cap S$ we have $\|y - x_0\| \geqslant 1$. Similarly, for $2^{-n}y$ we find $x_n \in K \cap S$ such that $\|2^{-n}y - x_n\| \geqslant 1$. Necessarily, $d([2^{-n}y, y], x_n) \geqslant 1$. Let $x \in K \cap S$ be a limit point of the (x_n). Then $[0, y] \cap B^0(x, 1) = \varnothing$. Apply the Hahn–Banach theorem to get a separating hyperplane $\psi^{-1}(0)$, where $\|\psi\| = -\psi(x) = 1$.

Since $W = \{\varphi \in B^*;\ \varphi(x) = -1\}$ is an extremal subset of B^*, $\psi \in \overline{\text{conv}}^{w^*} \text{ext } W = \overline{\text{conv}}^{w^*}(\text{ext } B^* \cap W)$. If $\varphi(y) \geqslant 0$ for every $\varphi \in \text{ext } W$, then $\{\varphi \in W;\ \varphi(y) = 0\}$ is an extremal subset of W and contains some $\varphi \in \text{ext } W \subset \text{ext } B^*$.

Remark. The case of a linear subspace Y was proved in [8] in an indirect way (using Singer's characterization of ext $B(C(K, E)^*)$).

2.7. Proposition (Laurent–Tuan). *If Y is a convex subset and K is a compact subset of the normed space E, $y_0 \in Y$, and $r_0 = r(y_0, K)$, then $y_0 \in z_Y(K)$ if and only if there is $\varphi_0 \in \overline{\text{conv}}^{w^*}\{\varphi \in \text{ext } B^*;$ there exists an $x \in K$ such that $\varphi(x - y_0) = r_0\}$ such that $\varphi_0(y_0) = \max_{y \in Y} \varphi_0(y)$.*

Proof. Sufficiency is immediate, since for every $y \in Y$, $\varphi_0(y_0) = \varphi_0(y)$ implies $\varphi(y_0) \geqslant \varphi(y)$ for some $\varphi \in B^*$ satisfying $\varphi(x - y_0) = r_0$ for some $x \in K$, hence, $r(y, K) \geqslant \|x - y\| = \varphi(x - y) = \varphi(x - y_0) = r_0$.

For necessity, we may take $y_0 = 0$, $r_0 = 1$. Let $W = \overline{\text{conv}}^{w^*}\{\varphi \in \text{ext } B^*;$ there exists an $x \in K$ such that $\varphi(x) = 1\}$. If for no $\varphi_0 \in W$ we have $\varphi_0(y) \leqslant 0$ for all $y \in Y$ then, by w^*-compactness of W, there are $y_1, ..., y_n \in Y$ such that $\max_{i=1,...,n} \varphi(y_i) > 0$ for all $\varphi \in W$. We may assume $y_1, ..., y_n$ to be minimal with respect to this property. Let $W_i = \{\varphi \in W;\ \varphi(y_i) \leqslant 0\}$, $i = 1, ..., n$. Apply now the following lemma of Klee [10].

2.8. Lemma. *If a compact convex set W in a locally convex space is the union $\bigcup_{i=1}^n W_i$ of n closed convex sets, and if $\bigcap_{i \neq j} W_i \neq \varnothing$ for every $j = 1, ..., n$, then $\bigcap_{i=1}^n W_i \neq \varnothing$.*

Proof. Induction on n. If proved for $n - 1$, assume $\bigcap_{i=1}^n W_i = \varnothing$. Strictly separate the disjoint convex sets W_n and $\bigcap_{i=1}^{n-1} W_i$ by a closed hyperplane H. For every $j = 1, ..., n-1$, the convex set $\bigcap_{j \neq i < n} W_i$ intersects both W_n and $\bigcap_{i=1}^{n-1} W_i$, which lie on opposite sides of H, hence intersects H. We can apply the induction hypothesis to $W_i \cap H$, $i = 1, ..., n - 1$, and arrive at the contradiction $\bigcap_{i=1}^{n-1} W_i \cap H \neq \varnothing$.

Remark. (1) Proposition 2.7 was deduced in [11] from convex analysis (subdifferential method).

(2) The deduction of Proposition 2.7 from Corollary 2.6 is a particular case of the "minimax theorem" (cf., e.g., [15]). We would like to thank S. Hart from Tel-Aviv University for referring us to this method.

(3) Assuming Proposition 2.7, Corollary 2.6 is easily deduced by observing that each $\{\varphi \in B; \varphi(x-y_0)=r_0\}$, $x \in K$, is a w^*-compact extremal subset of B^*, and that $\{\varphi \in B^*$; there exists an $x \in K$ such that $\varphi(x-y_0)=r_0\}$ is w^*-compact.

(4) Example 2.1 shows that none of the characterizations above is valid in the noncompact case.

2.9. Corollary. *If Y is a closed convex subset and K a compact convex subset of the Hilbert space H, then $z_Y(K) \in P_Y(K)$.*

Proof. We may assume $0 = z_Y(K)$, $r_Y(K) = 1$. By Proposition 2.7 there is $z_0 \in \overline{\text{conv}}\,\{z \in \text{ext}\, B(H)$; there exists an $x \in K$ with $(x, z) = 1\}$, with $(y, z_0) \leqslant 0$ for all $y \in Y$. Since $K \subset B(H)$, $(x, z) = 1$ can happen only when $x = z$, therefore, $z_0 \in K$. Since $(y, z_0) \leqslant 0$ for all $y \in Y$, $0 = P_Y z$.

When Y is n-dimensional, Proposition 2.7 yields, by the Carathéodory and Krein–Milman theorems, $\varphi_0 = \sum_{i=0}^{n} \alpha_i \varphi_i$, where $\varphi_i \in \text{ext}\, B^*$, $\varphi_i(x_i - y_0) = r_0$ for some $x_i \in K$, $\alpha_i \geqslant 0$ and we have $\sum_{i=0}^{n} \alpha_i = 1$.

Laurent–Tuan [11] and Rozema–Smith [13] studied $z_Y(K)$ for Y of the form $\{y \in x_0 + V;\ \psi(y) \leqslant w(\psi)$ for all $\psi \in \Psi\}$, where V is a subspace of E, Ψ is a w^*-compact subset of E^*, and w is w^*-continuous on Ψ. By Corollary 2.6 if $y_0 \in z_Y(K)$ then for every $z \in x_0 + V$ we either have $\psi(z) > w(\psi)$ for some $\psi \in \Psi$ or $\varphi(z) \leqslant \varphi(y_0)$ for some $\varphi \in \text{ext}\, B^*$ for which $\varphi(x - y_0) = r(y_0, K)$ for some $x_0 \in K$. Repeating the argument of Proposition 2.7, we get Theorem 1.2 of [11]: if Y is of the form above then $y_0 \in Y$ is in $z_Y(K)$ iff there is $\varphi_0 \in V^{\perp} \cap \overline{\text{conv}}^{w^*}(\{\varphi \in \text{ext}\, B^*$; $\varphi(x - y_0) = r(y_0, K)$ for some $x \in K\} \cup \{\psi; \psi(y_0) = w(\psi)\})$. If $\dim V = n$ then we can take $\varphi_0 = \sum_{i=1}^{r} \alpha_i \varphi_i + \sum_{i=1}^{s} \beta_i \psi_i$, $\alpha_i, \beta_i \geqslant 0$, $\sum \alpha_i + \sum \beta_i = 1$, $r \geqslant 1$, $s \geqslant 0$, and $r + s \leqslant n + 1$.

Moreover, taking such a φ_0 corresponding to y_0 in the relative interior of $z_Y(K)$, it will do for all $y \in z_Y(K)$ [11].

3. Existence of Chebyshev Centers

In [3] the following was claimed: if Y, A are subsets of a normed space E such that every $x \in A$ has a nearest $y \in K$ and every $y \in K$ has a farthest $x \in A$ then $z_Y(A) \neq \varnothing$.

This is clearly false, e.g., by Garkavi's result [9]: for every maximal subspace Y in a nonreflexive Banach space E there are $x \in E$ and an equivalent renorming of E such that under the new norm Y is proximinal but $z_Y(0, x) = \varnothing$.

In a recent paper on M-ideals by Yost [16], we found the following relevant result:

3.1. EXAMPLE (Yost). Let X be a strictly convex but not uniformly convex normed space, then $z_{c_0(E)}(u, v) = \varnothing$ for some $u, v \in l_\infty(E)$.

Indeed, if $x_n, y_n \in S_E$ are such that $\|x_n - y_n\| > 2\varepsilon > 0$ but $\delta_n \equiv \frac{1}{2}\|x_n + y_n\| \nearrow 1$, define $u, v \in l_\infty(E)$ by: $u(n) = x_n/\delta_n$, $v_n(n) = -y_n/\delta_n$, and $w_m \in c_0(E)$ by: $w_m(n) = (u(n) + v(n))/2$ for $n \leqslant m$, $w_m(n) = 0$ for $n > m$. Then

$$r(w_m, [u, v]) = \max(\max_{n \leqslant m} \tfrac{1}{2}\|u(n) - v(n)\|, \sup_{n>m} |u(n)|, \sup_{n>m} |v(n)|) = \delta_m^{-1} \to 1.$$

Hence, $r_{c_0(E)}(u, v) \leqslant 1$. But 1 cannot be attained since, by strict convexity of E, $\|w - u\| = \|w - v\| = 1$ has the only solution $w = (u + v)/2$ which, by our choice, is not in $c_0(E)$. Observe that $c_0(E)$ is an M-ideal in $l_\infty(E)$ (in particular, it is proximinal).

Yost shows that every $L_1(\mu)$ space and every WCG space (in particular, every separable normed space) can be equivalently renormed to be strictly convex but not even locally uniformly convex. He suggests that "there is no (infinite-dimensional) space for which every equivalent strictly convex norm is already locally uniformly convex." This is trivially false because of the existence of nonstrictly convexifiable spaces, e.g., $l_\infty(\Gamma)$ for uncountable Γ [7, p. 160], so that the right question should be: "Is every infinite dimensional locally uniformly convex space isomorphic to a strictly convex nonlocally uniformly convex one?" However, for our purpose we can state

3.2. PROPOSITION. *Every infinite dimensional normed space* $(E, \|\cdot\|)$ *has an equivalent norm under which* $z_{c_0(E)}(x, y) = \varnothing$ *for some* $x, y \in l_\infty(E)$.

Proof. Let E_0 be any infinite dimensional separable subspace of E, and let $\|\cdot\|_0$ be an equivalent strictly convex nonuniformly convex norm on E_0, with $\|\cdot\|_0 < \|\cdot\|$. Let B_1 be the closed convex hull of the $\|\cdot\|$-sphere S and the $\|\cdot\|_0$-sphere S_0, $\|\cdot\|_1$, the corresponding (equivalent) norm on E. The points of the $\|\cdot\|_1$-sphere can be represented by $z = \lambda x_0 + (1 - \lambda)x$, $\lambda \in [0, 1]$, $x_0 \in S_0$, $x \in S$. If $z \in B_1(0, 1) \cap B_1(2u, 1)$, $u \in S_0$, then because of strict convexity of $\|\cdot\|_0$ we must have $x_0 = u$, i.e., $z = \lambda u + (1 - \lambda)x$ and, similarly, $2u - z = \mu u + (1 - \mu)y$, $\lambda, \mu \in [0, 1]$, $x, y \in S$. But then $u = ((1 - \lambda)x + (1 - \mu)y)/(2 - \lambda - \mu) \in \operatorname{conv}[x, y]$, hence, $1 = \|u\|_1 = \|u\|_0 < \|u\| \leqslant 1$—a contradiction, unless $\lambda = \mu = 1$. Therefore, for $x_0, y_0 \in S_0$ we have $B_1(x_0, \|x_0 - y_0\|/2) \cap B_1(y_0, \|x_0 - y_0\|/2) = \{(x_0 + y_0)/2\}$. Take $x_n, y_n \in S_0$ with $\|x_n + y_n\|_1 = \|x_n + y_n\|_0 \to 2$, $\|x_n - y_n\|_1 = \|x_n - y_n\|_0 > 2\varepsilon$, and define $u, v \in l_\infty(E, \|\cdot\|_1)$ as in Yost's construction.

4. Lipschitz Continuity of the Chebyshev Center Map

Borwein and Keener [5] studied the Lipschitz constants $\mu(\mathscr{F}_i) = \sup\{\|z_A(A) - z_G(G)\|/h(A,G); (A,G) \in \mathscr{F}_i\}$, where $\mathscr{F}_i$, $i = 1, 2, 3$, are the following families of pairs of closed convex bounded sets in a fixed normed space E: $\mathscr{F}_1 = \{(A,G); B(z_A(A), r_A(A)) \cap B(z_G(G), r_G(G)) = \varnothing$, $\mathscr{F}_2 = \{(A,G); z_A(A) \notin G$ and $z_G(G) \notin A\}$, $\mathscr{F}_3 = \{(A,G); A \cap G = \varnothing\}$ (E is assumed to have unique self centers, e.g., a reflexive uced Banach space). They showed that if $\dim E \geqslant 2$ then $\frac{1}{2}(1+\sqrt{5}) \leqslant \mu(\mathscr{F}_1) \leqslant 2 \leqslant \mu(\mathscr{F}_3)$ and $\mu(\mathscr{F}_2) = \infty$, and asked if $\mu(\mathscr{F}_3) = 2$ for all E.

4.1. Proposition. *If* $\dim E \geqslant 3$ *then* $\mu(\mathscr{F}_3) = \infty$.

Proof. Let F be any 2-dimensional subspace of E, $x \in E \backslash F$, $\|x\| < 1$. Given any n, there are A, $H \subset F$ convex, closed, and bounded so that $h(A, H) = 1$ and $\|z_A(A) - z_H(H)\| > n + 1$ (since in F $\mu(\mathscr{F}_2) = \infty$). *Let* $G = A + x$. *Then* $\|z_A(A) - z_G(G)\| > n$ while $h(A, G) < 2$.

Borwein and Keener observed also that if one replaces the relative centers $z_A(A)$, $z_G(G)$ by the absolute centers $z(A) = z_E(A)$, $z(G) = z_E(G)$ and the relative radii $r_A(A)$, $r_G(G)$ by the absolute radii $r(A) = r_E(A)$, $r(G) = r_E(G)$, then the corresponding Lipschitz constants $\mu(\hat{\mathscr{F}}_i)$ $(i = 1, 2)$, still satisfy $\frac{1}{2}(1+\sqrt{5})) \leqslant \mu(\hat{\mathscr{F}}_1)$, $\mu(\hat{\mathscr{F}}_2) = \infty$. We show now that there is no upper bound for $\mu(\hat{\mathscr{F}}_1)$.

4.2. Example. E is the $(2n+1)$-dimensional space l_p^{2n+1}, $p > 1$, with the standard unit vector basis $(e_i)_{i=1}^{2n+1}$, $e = (1,..., 1)$, $A = \operatorname{conv}\{\sum_{i \in J} e_i; J \subset \{1,..., 2n+1\}, |J| = n\}$, $G = (1+\varepsilon)e - A$. Since A is invariant under the permutation isometries, we must have $z(A) = te$ for some constant t, and a simple computation shows that $t = (1 + ((n+1)/n)^{1/(p-1)})^{-1}$ and $r(A) = (n(1-t)^p + (n+1)t^p)^{1/p}$. Clearly, $z(G) = (1 + \varepsilon - t)e$ and $r(G) = r(A)$. In order to keep the Chebyshev balls disjoint, it suffices that $(1 + \varepsilon - 2t)(2n+1)^{1/p} > 2(n(1-t)^p + (n+1)t^p)^{1/p}$. For such ε, we have

$$\mu(\hat{\mathscr{F}}_1) \geqslant \frac{(1+\varepsilon-2t)(2n+1)^{1/p}}{(2n\varepsilon^p + (1+\varepsilon)^p)^{1/p}}.$$

Taking p almost 1, t and then ε can be made arbitrarily small, and the right-hand side arbitrarily close to $2n + 1$.

To get a space (infinite dimensional) with $\mu(\hat{\mathscr{F}}_1) = \infty$, take the l_2-direct sum $(\sum_{n=0}^{\infty} \oplus l_{p_n}^{2n+1})_2$, where p_n are chosen close enough to 1.

Problem. Is $\mu(\hat{\mathscr{F}}_1) \leqslant \dim E$ the right upper bound?

Borwein and Keener [5] showed also that if E is Hilbert space then $\mu(\mathscr{F}_1) = (\sqrt{5}+1)/2$. Their proof makes use of their Proposition 2, which is valid only for compact sets. Their theorem concerning $\mu(\mathscr{F}_1)$, however, remains true.

4.3. PROPOSITION. *If E is Hilbert space then* $\mu(\mathscr{F}_1) = (\sqrt{5}+1)/2$.

Proof. Let F_1, F_2 be bounded subsets of E. Without loss of generality, we may assume that $F_1 \subset B(0, a-\varepsilon)$ and $F_2 \subset B(x, 2-a)$, where $\|x\| = 2$, $0 < \varepsilon < a \leqslant 1$ (symmetry), $0 = z(F_1)$, $x = z(F_2)$. By Corollary 2.5 there is a $z \in F_2 \backslash B(x, 2-a-\varepsilon)$ such that $(x-z, x) \leqslant \varepsilon$. We then have $d(z, F_1) \geqslant d(z, B(0, a-\varepsilon)) = \sqrt{(2-a-\varepsilon)^2 + 4 - 2\varepsilon} - (a-\varepsilon)$ (since $(z/\|z\|)(a-\varepsilon)$ is the best approximation of z in $B(0, a-\varepsilon)$ and $\|z\|^2 \geqslant (2-a-\varepsilon)^2 + 4 - 2\varepsilon$). But the last function attains its minimum at $a = 1$, from which the proposition follows.

5. Semicontinuity of the Chebyshev Center Map

While the Chebyshev center map $F \to z(F)$ is known to be locally uniformly continuous in a certain class of normed spaces containing the uniformly convex and the $C_\sigma(\Omega)$ spaces [1, 4], examples where $F \to z(F)$ is not lower semicontinuous were given in [2]. Such is the case in the infinite dimensional $L_1(\mu)$ spaces, but it can happen even in 3-dimensional spaces.

A general condition for upper semicontinuity of $F \to z_Y(F)$ on the class of compact subsets is given in [6, 13, 14]: E has property (H) (i.e., $S \ni x_n \to^w x \in S \Rightarrow x_n \to x$) and Y is boundedly weakly sequentially compact, or: E is a dual space having property (H^*) (analogous, with w^* instead of w), and F is boundedly w^*-sequentially compact. No examples are given in the literature to see the necessity of any of the conditions. The following is an example where $F \to z(F)$ is not usc, even on the family of pairs $\{x, y\}$.

5.1. EXAMPLE. Consider the 3-dimensional space E_n^3 whose unit ball is the convex hull of the 18 points: $(\pm 1, \pm 1, \pm 1)$, $(\pm 1/n, \pm 1, \pm(1+1/n))$, $(0, 0, \pm(1+2/n))$.

The norm can be computed to be given by

$$\|(\xi, \eta, \zeta)\|_n = \max\left(|\xi|, |\eta|, \frac{|\xi| + n|\zeta|}{n+2}, \frac{|\eta| + n|\zeta|}{n+2}, \frac{|\xi| + (n-1)|\zeta|}{n}\right).$$

For every $|t| \leqslant 1$ we have $\|(0, 0, 1+2/n)\| = \|(1/n, t, 1+1/n)\| = 1$, hence, $z(\{\pm(0, 0, 1+2/n)\}) = \{(0, 0, 0)\}$, $z(\{\pm(1/n, 0, 1+1/n)\}) = \{(0, t, 0); |t| \leqslant 1\}$, while $\|(1/n, 0, 1+1/n) - (0, 0, 1+2/n)\| = \|(1/n, 0, -1/n)\| = 1/n$.

Now take $E=(\sum_{n=1}^{\infty}\oplus E_n^3)_\infty$ and consider the center map at $F_0=\{e,-e\}$, where $e(n)=(0,0,1+2/n)$. Clearly, $z(F_0)=0$. Let $F_m=\{e_m,-e_m\}$, where $e_m(n)=e(n)$ for $n\neq m$, $e_m(m)=(1/m,0,1+1/m)$. Then $z(F_m)=\{e_{m,t};|t|\leqslant 1\}$, where $e_{m,t}(n)=0$ for $n\neq m$, $e_{m,t}(m)=(0,t,0)$, and $h(F_0,F_m)=1/m$ while $h(z(F_0),z(F_m))=1$.

Example 5.2 shows that the Chebyshev center map need not have a continuous selection, even in the 3-dimensional case, i.e., one cannot select a single element $\varphi(A)\in z(A)$ such that $A\to\varphi(A)$ is continuous.

5.2. EXAMPLE. Let the unit ball of a norm $\|\cdot\|$ in E_3 be given by $\operatorname{conv}\{(0,\pm 1,0),\ (\pm 1,0,\pm 1)\}$ (8 points). Let $G_n=\{(-1+1/n,1/n,0),$ $(1-1/n,-1/n,0),\ (-1,0,1),\ (1,0,1)\}$. $F_n=\{(-1+1/n,\ 1/n,\ 1),\ (1-1/n,$ $-1/n,\ 1),\ (-1,\ 0,\ 0),\ (1,\ 0,\ 0)\}$. Then $h(G_n,F)\to 0$ and $h(F_n,F)\to 0$, where $F=\{(-1,0,0),\ (1,0,0),\ (-1,0,1),\ (1,0,1)\}$. Moreover, $z(F_n)=(0,0,0)$, $z(G_n)=(0,0,1)$ for all $n\in\mathbb{N}$, and $z(F)=[(0,0,0),(0,0,1)]$, so that the mapping $A\to z(A)$ cannot have a continuous selection in $(E_3,\|\cdot\|)$.

6. Proximinality, Existence of Chebyshev Centers, and Isomorphisms

Clearly, proximinality or admitting Chebyshev centers is an isometric property which cannot be expected to be invariant under isomorphisms. In [2] there is an example of an isomorphic strictly convex renorming of $C[0,1]$ which fails to admit Chebyshev centers. Most striking in this direction is Garkavi's construction mentioned in Section 3. Somewhat more restrictive is the case of two subspaces F, G of the same space E which are isomorphic under an automorphism T of E onto itself. Franchetti and Cheney [8] asked whether, in the case $E=C(\Omega)$, proximinality of F implies that of G. Except for the trivial cases when F is reflexive or T an isometry, every positive result is somewhat surprising, and such is their Theorem 4.8 (F of finite codimension, T a multiplication by an invertible $f\in C(\Omega)$).

6.1. EXAMPLE. In $E=c=c(\mathbb{N}^*)$, let $F\equiv\{x\in c_0; x(1)=0\}$, $G\equiv\{x\in c_0;\ \varphi(x)=0\}$ where $\varphi\equiv(1/2^n)_{n=1}^{\infty}\in l_1=c_0^*$. F is proximinal ($P_F(x)(1)=0$, $P_F(x)(n)=x(n)-\lim x$ for $n>1$), but G is not (even in c_0). $Tx(1)=x(1)-2\zeta(x)$, $Tx(n)=x(n)$ for $n\geqslant 2$ is an isomorphism of c onto itself carrying F onto G.

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